國立中正大學
108 學年度碩士班招生考試
試 題

[第 2 節]

系所組別	數學系統計科學
科目名稱	機率與統計

—作答注意事項—

※作答前請先核對「試題」、「試卷」與「准考證」之系所組別、科目名稱是否相符。

1. 預備鈴響時即可入場，但至考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。
2. 考試開始鈴響時，即可開始作答；考試結束鈴響畢，應即停止作答。
3. 入場後於考試開始 40 分鐘內不得離場。
4. 全部答題均須在試卷（答案卷）作答區內完成。
5. 試卷作答限用藍色或黑色筆（含鉛筆）書寫。
6. 試題須隨試卷繳還。

(15%) 1. Assume that $X_1, \ldots, X_n$ are independent and identically distributed (iid) uniform random variables U$(0, a)$, for any constant $a > 0$, and let $X_{(1)}, \ldots, X_{(n)}$ denote the order statistics.

(a)(5%) Find the probability density distribution (pdf) of $X_{(k)}$, for $1 \le k \le n$.

(b)(10%) Find the pdf of $M = [X_{(1)} + X_{(n)}]/2$.

(15%) 2. Assume that $X_i \sim \text{Exp}(\lambda_i)$ with pdf $f(x) = \lambda_i e^{-\lambda_i x}, x \ge 0$, for $i = 1, \ldots, n$. If $\{X_i\}_{i=1}^n$ are mutually independent and let $Z = \min(X_1, \cdots, X_n)$, then

(a)(5%) show that $Z \sim \text{Exp}(\sum_{i=1}^n \lambda_i)$.

(b)(10%) show that $P(Z = X_k) = \lambda_k / \sum_{i=1}^n \lambda_i$, for $k = 1, \cdots, n$.

(15%) 3. Assume that $X_1, \ldots, X_n \overset{iid}{\sim} \text{Exp}(1)$. For any x,

(a)(5%) show that $P\left(\frac{(\sum_{i=1}^n X_i) - n}{\sqrt{n}} \le x\right) \approx \Phi(x)$, where $\Phi(\cdot)$ is the cumulative distribution function (cdf) of the standard normal distribution.

(b)(10%) based on (a) to show that the Stirling's formula (hint: to differentiate and take $x = 0$):

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

(15%) 4. Let $X_1, \ldots, X_n$ be random samples from a Pareto distribution with pdf

$$f(x|\theta, \nu) = \frac{\theta \nu^\theta}{x^{\theta+1}} I_{[\nu,\infty)}(x), \quad \theta > 0, \quad \nu > 0.$$

(a)(5%) Find the maximum likelihood estimators (MLEs) of θ and ν.

(b)(10%) For hypothesis testing $\underline{H_0 : \theta = 1,\ \nu \text{ unknown}}$ versus $\underline{H_1 : \theta \neq 1,\ \nu \text{ unknown}}$, show that the likelihood ratio statistic is

$$\lambda(\mathbf{x}) = \left(\frac{T}{n}\right)^n e^{n-T}, \quad \text{where } T = \ln\left[\frac{\prod_{i=1}^n X_i}{(\min_i X_i)^n}\right].$$

(20%) 5. Suppose that $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{N}(\theta, \theta^2)$, $\theta > 0$, and denote $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

(a)(10%) Show that for any number a the estimator $a\bar{X} + (1-a)(cS)$ is an unbiased estimator of θ, where $c = \frac{\Gamma((n-1)/2)}{\Gamma(n/2)}\sqrt{\frac{n-1}{2}}$.

(b)(10%) Find the value of a such that the estimator $a\bar{X} + (1-a)(cS)$ can achieve the minimum mean squared error (MSE).

(20%) 6. Let $X_1, \ldots, X_n$ be independent with pdfs

$$f_{X_i}(x|\theta) = \begin{cases} e^{i\theta - x} & , \quad x \geq i\theta \\ 0 & , \quad \text{otherwise} \end{cases}.$$

(a)(5%) Prove that $T = \min_i (X_i/i)$ is a sufficient statistic for θ.

(b)(5%) Prove that $Y = (T - \theta)$ is a pivotal quantity.

(c)(10%) Based on T, find the shortest $(1-\alpha)$ confidence interval for θ of the form $[T+a, T+b]$.