(15%) 1. Evaluate the following integral and limit:

(5%) (a) $\int_2^3 \frac{x+5}{x^2+x-2}\, dx = ?$

(10%) (b) $\lim\limits_{x \to (\pi/2)^-} (\tan x)^{\cos x} = ?$

(10%) 2. Find the value of c such that $\sum\limits_{n=2}^{\infty} (1+c)^{-n} = 2$.

(15%) 3. If n is a positive integer, **prove that** (hint: use improper integral and integration by parts)

$$\int_{-\infty}^{0} x^n e^x \, dx = (-1)^n n!.$$

(15%) 4. Use **method of Lagrange multipliers** to find the maximum of $\sum\limits_{i=1}^{2} x_i y_i$ subject to the constraints $\sum\limits_{i=1}^{2} x_i^2 = 1$ and $\sum\limits_{i=1}^{2} y_i^2 = 1$.

(15%) 5. Find the eigenvalues and the corresponding eigenvectors of the operator $T : \boldsymbol{R}_2[x] \to \boldsymbol{R}_2[x]$ defined by $T(f(x)) = (f(x) + f(-x))/2$.

(15%) 6. Let A=$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^t A P = D$.

(15%) 7. Let $A_{n\times n}$ be the $n \times n$ matrix defined by

$$A_{n\times n}(i,j) = \begin{cases} 3 & \text{if } i = j \\ 2 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5%) (a) Write down $A_{5\times 5}$.

(5%) (b) Let a_n be the determinant of $A_{n\times n}$. **Show that** $a_n = 3a_{n-1} - 4a_{n-2}$, $n \geq 3$.

(5%) (c) Use part(b) to find the determinant of $A_{5\times 5}$.