1. Let X_1 and X_2 be two independent random variables, each having an exponential distribution with parameter λ, i.e. the probability density function (pdf) is

$$f(x_i, \lambda) = \lambda e^{-\lambda x_i}, x_i \geq 0, i = 1, 2$$

   (a) (10 points) Let $Z = \frac{X_1}{X_1+X_2}$, find the pdf of Z,
   (b) (10 points) Let $Z = \min(X_1, X_2)$, find the variance of Z

2. (10 points) Suppose the joint moment generating function (mgf), $M(t_1, t_2)$, exists for the continuous random variables X_1 and X_2. Show that X_1 and X_2 are independent if and only if

$$M(t_1, t_2) = M(t_1, 0)M(0, t_2);$$

that is, the joint mgf is identically equal to the product the marginal mgfs.

3. The Central Limit Theorem.

   (a) (10 points) State and prove the Central Limit Theorem (CLT)
   (b) (10 points) Use the CLT to show that

$$\lim_{n\to\infty} \int_0^n \frac{2}{(n-1)!} e^{-t} t^{n-1} dt = 1$$

   Please verify your answer clearly.

4. Let $X_1, \ldots, X_n$ be independent identical distribution (i.i.d) from $U(0, \theta), \theta > 0$.
   (a) (7 points) Show that $e \times (\prod_{i=1}^n X_i)^{1/n}$ converges to θ in probability.
   (b) (4 points) Find the maximum likelihood estimator (MLE) of θ.
   (c) (4 points) Find the sufficient statistic of θ.

5. Let $X_1, \ldots, X_n$ be i.i.d from Poisson(λ), i.e. the probability mass function is

$$P(X_i = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 1, 2, \ldots$$

   (a) (10 points) Find the uniformly minimum variance unbiased estimator (U.M.V.U.E) of $e^{-\lambda}$
   (b) (10 points) Let $n = 1$, find the U.M.V.U.E of $e^{-\lambda}$ and show that the U.M.V.U.E does not attain the Cramer-Rao Lower Bound.

6. (15 points) Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be independent random samples from the normal distributions $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, respectively. Suppose that σ^2 is unknown. Derive the size α likelihood ratio test for testing $H_0 : \mu_1 = \mu_2$ against $H_1 : \mu_1 \neq \mu_2$.