This exam has 6 questions, for a total of 100 points.

1. Let $f(x) = e^{-x^2}$, find
   (a) (5 points)
   $$\int_{-\infty}^{\infty} f(x)\,dx,$$
   (b) (10 points) the maximum and minimum values of $f(x)$ on $[-2, 4]$,
   (c) (10 points) the Taylor series expansion of $f(x)$ at $x = 0$.

2. For $a > 0$ and $n > 0$, obtain the following values
   (a) (5 points) $\lim_{x\to\infty}(\frac{\log x}{x^n})$,
   (b) (10 points) $\lim_{x\to\infty}(\frac{a^x}{x^n})$.

3. (10 points) Suppose that $g(x)$ and $h(x)$ are such that $g^2(x)$ and $h^2(x)$ are integerable on $[a, b]$. Show that
   $$\Big[\int_a^b g(x)h(x)\,dx\Big]^2 \le \Big[\int_a^b g^2(x)\,dx\Big]\Big[\int_a^b h^2(x)\,dx\Big].$$

4. (10 points) Find
   $$\lim_{n\to\infty}\{\frac{n}{n^2} + \frac{n}{n^2+1} + \ldots + \frac{n}{n^2+(n-1)^2}\}.$$

5. Let $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$.
   (a) (10 points) Find the eigenvalues and the corresponding eigenvectors of $\mathbf{A}$.
   (b) (10 points) Find the eigenvalues and the corresponding eigenvectors of $\mathbf{A}^{100}$.

6. Using the following theorem to answer the questions.

   Theorem. *Let $\mathbf{A}$ be a symmetric matrix of order $n \times n$. There exists an orthogonal matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P\Lambda P}'$, where $\mathbf{\Lambda} = Diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\mathbf{A}$. The columns of $\mathbf{P}$ are the corresponding orthonormal eigenvectors of $\mathbf{A}$.*

   (a) (10 points) Find the trace and the determinant of $\mathbf{A}$, in terms of $\lambda_i, i = 1, \ldots, n$.
   (b) (10 points) If $\mathbf{A}$ is also a positive definite matrix, show that $\lambda_i > 0, i = 1, \ldots, n$.