國立中正大學
109 學年度碩士班招生考試
試 題
[第 2 節]

科目名稱	機率與統計
系所組別	數學系統計科學

—作答注意事項—

※作答前請先核對「試題」、「試卷」與「准考證」之系所組別、科目名稱是否相符。

1. 預備鈴響時即可入場，但至考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。
2. 考試開始鈴響時，即可開始作答；考試結束鈴響畢，應即停止作答。
3. 入場後於考試開始 40 分鐘內不得離場。
4. 全部答題均須在試卷（答案卷）作答區內完成。
5. 試卷作答限用藍色或黑色筆（含鉛筆）書寫。
6. 試題須隨試卷繳還。

(10%) 1. Let (X,Y) be a two-dimensional random variable. Show that

(1) $E[Y] = E\big[E[Y|X]\big]$.

(2) $\mathrm{var}[Y] = E\big[\mathrm{var}[Y|X]\big] + \mathrm{var}\big[E[Y|X]\big]$.

(18%) 2. Let $X_1, \cdots, X_n$ be a random sample from the density $f(\cdot;\theta)$, and let $S_1 = s_1(X_1, \cdots, X_n), \cdots, S_k = s_k(X_1, \cdots, X_n)$ be a set of jointly sufficient statistics. Let the statistic $T = t(X_1, \cdots, X_n)$ be an unbiased estimator at $\tau(\theta)$. Define T' by $T' = E[T|S_1, \cdots, S_k]$. Show that

(1) T' is a statistic, and it is a function of the sufficient statistics $S_1, \cdots, S_k$. Write $T' = t'(S_1, \cdots, S_k)$.

(2) $E_\theta[T'] = \tau(\theta)$; that is, T' is an unbiased estimator of $\tau(\theta)$.

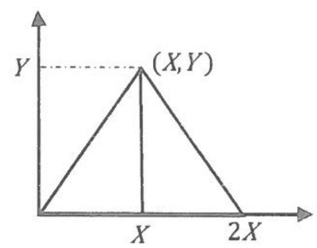
(3) $\mathrm{var}_\theta[T'] \le \mathrm{var}_\theta[T]$ for every θ, and $\mathrm{var}_\theta[T'] < \mathrm{var}_\theta[T]$ for some θ unless T is equal to T' with probability 1.

(18%) 3. Suppose that the joint probability density function of (X,Y) is given by

$$f_{X,Y}(x,y) = [1 - \alpha(1-2x)(1-2y)]I_{(0,1)}(x)I_{(0,1)}(y),$$

where the parameter α satisfies $-1 \le \alpha \le 1$.

(1) Prove or disprove X and Y are independent if and only if X and Y are uncorrelated.



(2) If (X,Y) has the joint density given above, pick α to maximize the expected area of the isosceles triangle indicated in the sketch.

(3) What is the probability that the triangle falls within the unit square with corners at (0,0), (1,0), (1,1), and (0,1)?

(24%) 4. Let $X_1, \cdots, X_n$ be a random sample from

$$f(x;\theta) = e^{-(x-\theta)}I_{(\theta,\infty)}(x), \text{ for } -\infty < \theta < \infty.$$

(1) Find a sufficient statistic.

(2) Find a maximum-likelihood estimator of θ.

(3) Find a method-of-moments estimator of θ.

(4) Using the prior density $g(\theta) = e^{-\theta}I_{(0,\infty)}(\theta)$, find the posterior Bayes estimator of θ.

(30%) 5. Let $X_1, \cdots, X_n$ be a random sample from $f(x;\theta) = \theta e^{-\theta x} I_{(0,\infty)}(x)$. Let θ_0 and θ_1 be known fixed numbers, and we assume that $\theta_1 > \theta_0 > 0$.

(1) Find the most powerful size-α test of $\boldsymbol{H}_0: \theta = \theta_0$ versus $\boldsymbol{H}_1: \theta = \theta_1$.

(2) Show that $\{f(x;\ \theta): \theta > 0\}$ has a monotone likelihood-ratio (MLR).

(3) Is there a uniformly most powerful size-α test of $\boldsymbol{H}_0: \theta \leq \theta_0$ versus $\boldsymbol{H}_1: \theta > \theta_0$? If so, what is it ?

(4) Find a generalized likelihood-ratio test of size α for testing $\boldsymbol{H}_0: \theta = \theta_0$ versus $\boldsymbol{H}_1: \theta \neq \theta_0$. And find the acceptance region $\mathrm{A}(\theta_0)$.

(5) Inverting the acceptance region in (4) to find a $1 - \alpha$ confidence interval for θ.