(24%) 1. Find the following limits:

(8%) (a) $\lim\limits_{n\to\infty}\left(\prod_{k=1}^{n} \sqrt[2^k]{7}\right)$

(8%) (b) $\lim\limits_{n\to\infty}(x^n + y^n)^{1/n}$ where x and y are positive real numbers independent of n

(8%) (c) $\lim\limits_{x\to 0^+} x^x$

(10%) 2. Evaluate $\int_0^1 \ln(x+1)\,dx$.

(16%) 3. Let $m_0 = 1$, $m_1 = 6$, and

$$m_{k+1} = \frac{m_{k-1} + (4k-1)m_k}{4k}$$

for all positive integer k. Find $\lim\limits_{n\to\infty} m_n$.

(10%) 4. Let $X = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Find the orthogonal projection matrix onto the column space of X.

(10%) 5. Let λ be an eigenvalue of the square matrix M. Show that λ^2 is an eigenvalue of M^2.

(10%) 6. Let $B = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$. Find the minimum value of $\frac{v^T B v}{v^T v}$ for any nonzero real vector $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, where v^T is the transpose of v.

(20%) 7. Let $x_0 = 1$ and $y_0 = -1$. Suppose that
$$x_n = 4x_{n-1} + y_{n-1} \quad \text{and} \quad y_n = -2x_{n-1} + 7y_{n-1}$$
for all positive integer n. Express each of x_n and y_n explicitly in terms of n.