(15%) 1. A random variable X has the probability mass function as follows

$$P(X=x)=\frac{-p^x}{x\ln(1-p)},\quad x=1,2,\cdots,\quad 0<p<1.$$

(a)(5%) Verify that this is exactly a probability mass function.

(b)(5%) Find the mean $\mathrm{E}(X)$.

(c)(5%) Find the moment generating function $\mathrm{M}_X(t)$.

Hint: Use a Taylor series for $\ln(1-p)$.

(15%) 2. Suppose that

$$N|Y=y\sim \mathrm{Po}(y)\quad,\quad Y\sim\Gamma(\alpha,\lambda).$$

where shape $\alpha\in\mathbb{N}$ and λ is the rate parameter.

(a)(5%) Find $\mathrm{E}(N)$.

(b)(5%) Find $\mathrm{Var}(N)$.

(c)(5%) Find $P(N=n)$.

(10%) 3. Assume that the sequence of random variables $\{X_n\}_{n=1}^{\infty}$ are independent and identically distributed (iid) having $P(X_n=1)=P(X_n=2)=1/2,\quad \forall n\geq 1$. What do the geometric averages converge to? That is,

$$\lim_{n\to\infty}\left(\prod_{i=1}^{n}X_i\right)^{1/n}=?$$

(30%) 4. Let $X_1,\ldots,X_n$ be a sample from Weibull(η,β_0) with pdf

$$f(x|\eta)=\frac{\beta_0 x^{\beta_0-1}}{\eta^{\beta_0}}\exp\left(-\frac{x^{\beta_0}}{\eta^{\beta_0}}\right),\quad x>0,\quad \eta>0,$$

where β_0 is a given value.

(a)(5%) Find the k^{th}-moment of Weibull(η,β_0).

(b)(5%) Find the method of moments estimator $\tilde{\eta}$ of η.

(c)(5%) Find the maximum likelihood estimator $\hat{\eta}$ of η.

(d)(5%) Construct the exact $100(1-\alpha)\%$ confidence interval for η.

(e)(10%) Find the sampling distribution distribution of $\hat{\eta}$.

(20%) 5. Let X be one observation from a distribution with

$$f(x|\theta) = \frac{\theta 2^{\theta}}{x^{\theta+1}}, \quad x > 2, \quad \theta > 0.$$

(a)(8%) Find the most powerful test for $H_0 : \theta = 2$ versus $H_1 : \theta = 1$ at $\alpha = 0.04$.

(b)(4%) Show that this family has an MLR (monotone likelihood ratio).

(c)(8%) Find the uniformly most powerful test for $H_0 : \theta = 2$ versus $H_1 : \theta > 2$ at $\alpha = 0.36$.

(10%) 6. Toss three coins 72 times and record the number of head (H). The results are shown in following table. Test the claim that these coins are fair and randomly tossed at $\alpha = 0.05$. ($\chi^2_{0.05}(1) = 3.841$, $\chi^2_{0.05}(2) = 5.991$, $\chi^2_{0.05}(3) = 7.815$, $\chi^2_{0.05}(4) = 9.488$)

Number of H	0	1	2	3
Observed frequency	12	25	25	10