國立中正大學
111 學年度碩士班招生考試
試 題
[第2節]

科目名稱	線性代數
系所組別	數學系應用數學

—作答注意事項—

※作答前請先核對「試題」、「試卷」與「准考證」之系所組別、科目名稱是否相符。

1. 預備鈴響時即可入場，但至考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。
2. 考試開始鈴響時，即可開始作答；考試結束鈴響畢，應即停止作答。
3. 入場後於考試開始 40 分鐘內不得離場。
4. 全部答題均須在試卷（答案卷）作答區內完成。
5. 試卷作答限用藍色或黑色筆（含鉛筆）書寫。
6. 試題須隨試卷繳還。

1. Let P_2 be the set of polynomials with real coefficients of degree less than or equal to 2. Define a transformation $T : P_2 \to P_2$ as

$$T(\alpha + \beta x + \gamma x^2) = (2\alpha + \beta + \gamma) + (2\alpha + \beta - 2\gamma)x - (\alpha + 2\gamma)x^2 \quad \text{for any } \alpha, \beta, \gamma \in \mathbb{R}$$

(a) (8 points) Is T a linear transformation? Is T an isomorphism?

(b) (8 points) Find the matrix of T relative to the basis $B = \{1 - x^2, 1 + x, 2x + x^2\}$.

(c) (8 points) Find the eigenspaces for T.

2. Let matrix A be

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

Let $V = \mathbb{R}^3$. We define the map $* : V \times V \to \mathbb{R}$ by $u * v = u^T A v$ for all $u, v \in V$.

(a) (8 points) Prove that $*$ is an inner product on V .

(b) (8 points) Use the inner product from above and the Gram-Schmidt orthogonalization process to find an orthonormal basis for V .

3. (12 points) Find the general solution to the system of ordinary differential equations

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{u}$$

where $\mathbf{u} = [u_1, u_2, u_3]^T$.

4. (12 points) Let A be the matrix

$$A = \begin{bmatrix} 3 & 3 & -4 \\ -4 & -3 & 5 \\ 2 & 4 & 0 \end{bmatrix}$$

Evaluate $4A^5 - 5A^4 - 50A^3 - 76A^2 - 10A + 50I$.

5. Let A, B, and C represent three real $n \times n$ matrices, where A and B be symmetric positive definite (spd) and C be invertible. Prove that each of the following is spd. (A real symmetric matrix A is positive definite if and only if $x^T A x > 0$ for all real n-dimensional vectors $x \neq 0$, or equivalently, if all its eigenvalues are real and positive.)

   (a) (4 points) A^{-1}

   (b) (4 points) $A + B$

   (c) (4 points) $C^T A C$

   (d) (4 points) $A^{-1} - (A + B)^{-1}$

6. Prove or disprove the following statement.

   (a) (5 points) All eigenvalues of a nilpotent matrix are zero. (A square matrix N is called nilpotent if $N^m = 0$ for some positive integer m).

   (b) (5 points) Let A and B be 2×2 real matrices. Then $AB = O$ implies $A = O$ or $B = O$, where O is the zero matrix.

   (c) (5 points) Let A be an $m \times n$ matrix with linearly independent columns. Then AA^T is an invertible matrix.

   (d) (5 points) Let w be a vector in $\mathbb{R}^n$ of length 1, the $H = I - 2ww^T$ is a projection matrix.