國立中正大學
109學年度碩士班招生考試
試題
[第2節]

科目名稱	線性代數
系所組別	數學系應用數學

—作答注意事項—

※作答前請先核對「試題」、「試卷」與「准考證」之系所組別、科目名稱是否相符。

1. 預備鈴響時即可入場，但至考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。
2. 考試開始鈴響時，即可開始作答；考試結束鈴響畢，應即停止作答。
3. 入場後於考試開始40分鐘內不得離場。
4. 全部答題均須在試卷（答案卷）作答區內完成。
5. 試卷作答限用藍色或黑色筆（含鉛筆）書寫。
6. 試題須隨試卷繳還。

1. Let A be the matrix

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$$

 (a) (10 points) Find the eigenvalues (and their algebraic and geometric multiplicities) and their eigenspaces.

 (b) (5 points) Find A^n, where n is a positive integer.

2. Define linear transformation $S \in \mathcal{L}(\mathbb{R}^4)$ by $S : (w_1, w_2, w_3, w_4)^T \to (0, w_2 + w_4, w_3, w_4)^T$.

 (a) (10 points) Determine the minimal polynomial of S.

 (b) (5 points) Determine the characteristic polynomial of S.

 (c) (5 points) Determine the Jordan form of S.

3. Let P_3 be the space of all real polynomials of degree at most 3. Equip P_3 with the inner product $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$.

 (a) (10 points) Apply the Gram-Schmidt process to the basis $\mathfrak{B} = \langle 1, x, x^2, x^3 \rangle$.

 (b) (5 points) Find the orthogonal complement of the subspace of scalar polynomials.

4. Let L be the linear operator on P_2 (the space of all real polynomials of degree at most 2), defined by

$$L(p(x)) = xp'(x) + p''(x)$$

 (a) (5 points) Find the matrix A representing L with respect to the basis $\mathfrak{B} = \langle 1, x, x^2 \rangle$.

 (b) (5 points) Find the matrix B representing L with respect to the basis $\mathfrak{B}' = \langle 1, x, 1 + x^2 \rangle$.

 (c) (5 points) Find a matrix S such that $B = S^{-1}AS$.

 (d) (5 points) If $p(x) = a_0 + a_1 x + a_2(1 + x^2)$, calculate $L^n(p(x))$.

5. (10 points) An $n \times n$ matrix A is Hermitian if $A = A^\star$, where $A^\star$ is the conjugate transpose of A. It is unitary if $AA^\star = I_n$, where I_n is the identity matrix. Let A and B be $n \times n$ Hermitian matrices over $\mathbb{C}$. If A is positive definite, i.e. all eigenvalues of A are positive, show that there exists an invertible matrix P such that $P^\star AP = I_n$ and $P^\star BP$ is diagonal.

6. Prove that the following is true or provide a counterexample if it is false.

(a) (5 points) Let $T : V \to W$ be a linear transformation. Assume that $\dim V = n$, $\dim W = m$ and $n < m$. Then T is onto.

(b) (5 points) Let A denote a real $n \times n$ symmetric matrix. Then there exists a real symmetric matrix B such that $A = B^2$.

(c) (5 points) If A and B are Hermitian matrices, then so is AB.

(d) (5 points) Let A and B be invertible $n \times n$ matrices. Then, if $A+B$ is invertible, we have $A^{-1}+B^{-1}$ is invertible.