# 國立中正大學 110 學年度碩士班招生考試試題

科目名稱：微積分 本科目共 2 頁第 1 頁
系所組別：數學系
數學系應用數學

**1.** (10 pts) Evaluate the following limits

**a.** (5 pts) $\lim_{x\to 0} \frac{x^2}{2}\Big(\frac{x}{\sin x + x} - \frac{x}{\sin x - x} - 2\Big)$.

**b.** (5 pts) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\sqrt{n}\sqrt{n+i}}$.

**2.** (10 pts) Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

**a.** (2 pts) Use the definition of derivative to evaluate $f'(0)$.

**b.** (5 pts) Show that $f$ has derivatives of all orders that are defined on $\mathbb{R}$.

**c.** (3 pts) Show that $f(x)$ is not an analytic function on $\mathbb{R}$.

**3.** (10 pts) Let $n \geq 2$ be positive integer, and $f(x) = [\![ x ]\!]$ be the floor function.

**a.** (5 pts) Show that

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \dfrac{2^{n-1}\big( [\![\frac{n}{2}]\!]!\big)^2}{n!}, & \text{if } n \text{ is odd} \\ \dfrac{n!\pi}{2^{n+1}\big( [\![\frac{n}{2}]\!]!\big)^2}, & \text{if } n \text{ is even} \end{cases}$$

**b.** (5 pts) Find the exact length of the polar curve $r = \sin^n\left(\frac{\theta}{n}\right)$ for $n \in \mathbb{N}$.

**4.** (10 pts) Evaluate the following integrals

**a.** (5 pts) $\int_2^{\infty} \frac{dx}{x^2\sqrt{4x+1}}$. **b.** (5 pts) $\int_2^{\infty} \frac{dx}{x\sqrt{x^2-4}}$.

**5.** (10 pts)

**a.** (5 pts) Find the Maclaurin series for the function $f(x) = \ln(x + \sqrt{x^2+1})$.

**b.** (5 pts) Determine the interval of convergence for the above series.

**6.** (10 pts) A plane curve is parametrized by $\mathbf{r}(t) = (\cos t + t\sin t, \sin t - t\cos t)$, $t > 0$.

**a.** (6 pts) Compute the unit tangent vector $\mathbf{T}(t)$, the unit normal vector $\mathbf{N}(t)$, and the curvature $\kappa(t)$.

**b.** (4 pts) Show that all centers of osculating circles, $\mathbf{r}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t)$, lie on a circle.

7. (10 pts) Let $f(x,y) = [\sin(x^2+y^2)]^{x-y}$.

   a. (5 pts) Show that $f$ has the same limiting value along every line through the origin.

   b. (5 pts) Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? If so, find the limit.

8. (10 pts)

   a. (6 pts) Let $\alpha > 0$. Find the minimum $f(x,y) = \alpha xy$ where $x, y > 0$, subject to the constraint $x^4 - x^2y^2 + y^2 = 0$.

   b. (4 pts) Find the equation of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ a, b > 0$ encloses the circle $x^2 + y^2 = 2y$ with minimal area.

9. (10 pts)

   a. (5 pts) Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

   b. (5 pts) Show that the double integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-(x-y)^2}}{1+(x+y)^2}\,dxdy = \frac{\pi\sqrt{\pi}}{2}.$$

   (Hint: Evaluate the improper integral by integrating over the square $S_a = \{(x,y) \mid -a \le x \le a, -a \le y \le a\}$ and taking the limit as $a \to \infty$.)

10. (10 pts) Is there a simple closed curve $C$ in the $xy$-plane which maximizes the integral $\displaystyle\oint_C y^3\,dx + (3x - x^3)\,dy$? If so, find the equation of the curve and the maximum value of the integral.