1. (30 pts.) Let $A = \begin{bmatrix} 0 & 0 & 1 & -3 & 2 \\ 2 & -1 & 4 & 2 & 1 \\ 4 & -2 & 9 & 1 & 4 \\ 2 & -1 & 5 & -1 & 5 \end{bmatrix}$.
   (a) (5 pts.) If P is any permutation, show that there is a nonzero vector x so that $(\mathrm{I}-\mathrm{P})\mathrm{x} = 0$.
   (b) (10 pts.) Find the factorization of $\mathrm{PA} = \mathrm{LU}$, where L is lower triangular and U is upper triangular.
   (c) (5 pts.) Find the rank of A and a basis for the row space of A.
   (d) (5 pts.) Find a basis for the column space of A.
   (e) (5 pts.) Find a basis for the nullspace of A.

2. (10 pts.) In $R^3 = \{(x, y, z) | x, y, z \in R\}$, we define the addition by

   $$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1, z_1 + z_2 + 1)$$

   (a) (4 pts.) Find the additive identity and the inverse of (x, y, z).
   (b) (6 pts.) Can we define a scalar multiplication with the above addition operation so that R^3 is a vector space? If your answer is yes, please give the definition of scalar multiplication.

3. (10 pts.) Apply Gram-Schmidt process to find the QR factorization for matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, where Q is orthogonal and R is upper triangular.

4. (10 pts.) Let $\mathrm{B} = \{(1,1,-1), (1,-1,1), (-1,1,1)\}$ and $\mathrm{B}' = ((1,0,0), (0,1,0), (0,0,1)\}$ be bases for R^3, and let $A = \begin{bmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix}$ be the matrix for $\mathrm{T}: R^3 \to R^3$ relative to B. Let $[v]_B$ denote the coordinate of v relative to the basis B.
   (a) (4 pts.) Find $[v]_B$ and $[T(v)]_B$, where $[v]_{B'} = [2 \quad 1 \quad 1]^T$.
   (b) (6 pts.) Find the matrix for T relative to B'.

5. (10 pts.) Find the general solution to $\frac{\mathrm{d}}{\mathrm{dt}}y = Ay$ for $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

6. (30 pts.) Determine whether each of the following statement is true or false. Prove it if it is true. Give a counterexample if it is false.
   (a) (6 pts.) If $\mathrm{P} = P^T P$, then P is a projection matrix.
   (b) (6 pts.) If we define $(\mathrm{S}+\mathrm{T})(\mathrm{v}) = \mathrm{S}(\mathrm{v}) + \mathrm{T}(\mathrm{v})$ where S and T are two linear transformations from vector space V to vector space W, then $\mathrm{rank}(\mathrm{S}+\mathrm{T}) \leq \mathrm{rank}(\mathrm{S}) + \mathrm{rank}(\mathrm{T})$.
   (c) (6 pts.) Let $\mathrm{S}: \mathrm{V} \to \mathrm{W}$ and $\mathrm{T}: \mathrm{W} \to \mathrm{U}$ be two linear transformations. If S is not one-to-one, then neither is $\mathrm{T} \circ \mathrm{S}$.
   (d) (6 pts.) An inner product space isomorphism preserves angles and distances.
   (e) (6 pts.) Let $\mathrm{C}[0, \pi]$ have the inner product $\langle f, g \rangle = \int_0^\pi f(x)g(x)dx$ and $f_k(x) = \cos kx,\ k = 0, 1, 2, \ldots,$ then f_m and f_n are orthogonal for $\mathrm{m} \neq \mathrm{n}$.