(請提供詳細計算或證明過程, 僅有答案而沒有過程得零分!)

1. (12 points) Consider the system of linear equations $Ax = b$ which is represented by the following augmented matrix:

$$\left[\begin{array}{cccc|c} 2 & 3 & -2 & 4 & 2 \\ -6 & -9 & 7 & -8 & -3 \\ 4 & 6 & -1 & 20 & 13 \end{array}\right],$$

where $x = [x_1, x_2, x_3, x_4]^T$. What is the rank of the coefficient matrix A? Does this linear system have solution? Why or why not? Find the solution(s) if exist.

2. (12 points) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$L(x) = (x^T v)w, \quad \text{where } v = [1, 2, -1]^T, \quad w = [2, 2, 1]^T.$$

Find the matrix of the linear transformation. Then, find bases for the range and the kernel of L.

3. (12 points) Find the matrix of the quadratic equation

$$3x^2 - 10xy + 3y^2 + 16\sqrt{2}y - 64 = 0.$$

Perform a rotation of axes to eliminate the xy-term in the quadratic equation. Find the equation of the rotated conic. Is it an ellipse? parabola? or hyperbola?

4. (12 points) Let V be a subspace of $\mathbb{R}^4$ spanned by vectors $v_1 = [1, 1, 0, 0]$, $v_2 = [2, 0, -1, 1]$, and $v_3 = [0, 1, 1, 0]$. Find orthogonal basis for the subspace V and the distance from the point $y = [0, 0, 0, 4]$ to the subspace V.

5. (12 points) Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

   Evaluate $P(A) = 3A^5 - 5A^4 - 15A^3 + 5A^2 + 10A - 5I$.

6. (12 points) Let A be a 3×3 matrix. Let x, y and z be linearly independent vectors in $\mathbb{R}^3$ and suppose that $Ax = [3, -1, 0]^T$, $Ay = [-2, 1, 2]^T$ and $Az = [1, 0, 2]^T$. Find the determinant of the matrix A.

7. (12 points) Let $C(\mathbb{R})$ be the vector space of all real-valued continuous functions. For what real value of α is the set
$$S_\alpha = \{f \in C(\mathbb{R}) : \ f(0) = \alpha\}$$
   a subspace of the vector space $C(\mathbb{R})$? Prove your answer.

8. (16 points, 8 points each) Determine if the following statements are true or false. Provide a proof if it is true or give any explanation/counterexample if it is false.

   (a) If an $n \times n$ matrix A has n distinct eigenvectors, then A is diagonalizable.

   (b) Matrix A is called skew-symmetric if $A^T = -A$. Suppose that n is an odd integer and let A be an $n \times n$ skew-symmetric matrix. The determinant of A is zero.