國立中正大學

108 學年度碩士班招生考試

試題

［第 2 節］

系所組別	數學系應用數學
科目名稱	線性代數

—作答注意事項—

※作答前請先核對「試題」、「試卷」與「准考證」之系所組別、科目名稱是否相符。

1. 預備鈴響時即可入場，但至考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。
2. 考試開始鈴響時，即可開始作答；考試結束鈴響畢，應即停止作答。
3. 入場後於考試開始 40 分鐘內不得離場。
4. 全部答題均須在試卷（答案卷）作答區內完成。
5. 試卷作答限用藍色或黑色筆（含鉛筆）書寫。
6. 試題須隨試卷繳還。

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There are five problems with 100 points. Show your work for partial credits.

Notations: Let $\mathbb{R}$ be the field of real numbers; $\mathbb{R}^n$ the vector space of dimension n over $\mathbb{R}$ and $\mathbb{R}^{m\times n}$ the family of all $m \times n$ matrices over $\mathbb{R}$.

1. (30 pts) Let $A = \begin{bmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & -2 & 4 & -2 & 0 \\ 3 & 11 & -4 & -1 & 6 \\ 2 & 5 & 3 & -4 & 0 \end{bmatrix}$.
   (a). Find the reduced row echelon form which is row equivalent to A. (10 pts)
   (b). According to (a), find bases for the row space and column space of A, respectively. (5 pts)
   (c). Find a basis for the null space of A. (5 pts)
   (d). Let the vector space $\mathbb{R}^{1\times 5}$ be endowed with the standard inner product.
   Find an orthonormal basis for the row space of A. (10 pts)
2. (10 pts) If $A \in \mathbb{R}^{3\times 3}$ satisfies $A^3 = O \neq A^2$, where $O \in \mathbb{R}^{3\times 3}$ is the zero matrix, show that there is an invertible $P \in \mathbb{R}^{3\times 3}$ such that $P^{-1} \cdot A \cdot P$ is an upper triangular matrix.
3. (10 pts) Suppose that $P \in \mathbb{R}^{m\times n}$ and $Q \in \mathbb{R}^{n\times m}$ are two matrices. Prove that the nonzero eigenvalues of $P \cdot Q$ are the same as those of $Q \cdot P$.
4. (20 pts) Let $\alpha \in \mathbb{R}$, $\vec{u}, \vec{v} \in \mathbb{R}^{n\times 1}$ and $A \in \mathbb{R}^{n\times n}$ be invertible. Define $B = \begin{bmatrix} A & \vec{u} \\ \vec{v}^T & \alpha \end{bmatrix}$.
   (a). Show that if $\alpha \neq \vec{v}^T A^{-1} \vec{u}$, then B is invertible. (10 pts)
   (b). If $\alpha = \vec{v}^T A^{-1} \vec{u}$, then $B\vec{x} = \vec{0}$ has nontrivial solutions. (10 pts)
5. (3 pts) (a). What is the definition of $tr(A)$, the trace of a square matrix A?
   (6 pts) (b). If A and B are both $n \times n$ real matrices, prove that $tr(AB) = tr(BA)$.
   (6 pts) (c). Find the trace of $I + A + A^2 + \cdots + A^{10}$ for $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$.
6. (15 pts) Prove or disprove the following statements.
   (a). If $A \in \mathbb{R}^{m\times n}$, then the null space of A is the same as the null space of $A^T \cdot A$. (5 pts)
   (b). If $A \in \mathbb{R}^{m\times n}$, then the row space of A is the same as the row space of $A^T \cdot A$. (5 pts)
   (c). If $A, B \in \mathbb{R}^{n\times n}$ such that $A \cdot B = O$, where $O \in \mathbb{R}^{n\times n}$ is the zero matrix, then $\text{rank}(A) + \text{rank}(B) \leqslant n$. (5 pts)