(3) (25 points) Remember that the addition and multiplication in the prime field $\mathbb{Z}_2$ are given by

+	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Let

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

be a matrix with entries in $\mathbb{Z}_2$.

(a) Find the characteristic polynomial of A.

(b) Give the algebraic multiplicity and the geometric multiplicity for each eigenvalue of A.

(c) Determine the Jordan form of A.

(d) Is A diagonalizable over $\mathbb{Z}_2$?

(4) (25 points) Let A be a square matrix over $\mathbb{C}$. Show that over $\mathbb{C}$, the matrix A is similar to A^t, the transpose of A.

Attentions!! In the following problems, $\mathbb{R}$ denote the field of real numbers, $\mathbb{C}$ denote the field of complex numbers, and $\mathbb{Z}_2 = \{0, 1\}$ denote the prime field of characteristic 2.

(1) (30 points) Let $V = \mathcal{F}([0, 2\pi], \mathbb{R})$ be the set of functions from $[0, 2\pi]$ to $\mathbb{R}$. Suppose you already know that V is a real vector space with respect to

$$(f+g)(x) \stackrel{\text{def}}{=} f(x) + g(x),$$
$$(cf)(x) \stackrel{\text{def}}{=} cf(x),$$

where f and $g \in V$ and $c \in \mathbb{R}$.

(a) Determine whether each of the following subsets is a subspace:

$$W_1 = \{f \in V : f \text{ is differentiable in } (0, 2\pi)\},$$
$$W_2 = \{f \in V : f \text{ is integrable over } [0, 2\pi]\},$$
$$W_3 = \{f \in V : f(0) = 0\}$$
$$W_4 = \{f \in V : f(0) \geq 0\}.$$

Be sure to give your reasons.

(b) Restrict the domains of $\sin x$, $\cos x = \sin(x + (\pi/2))$ and $\sin(x + \pi)$ to $[0, 2\pi]$ so that they belong in V. Are they linearly independent over $\mathbb{R}$?

(2) (20 points) Let $a_0, a_1, a_2, \ldots, a_n$ be $n+1$ distinct real numbers. Let

$$f_i = \frac{(x-a_0)(x-a_1)\cdots(x-a_{i-1})(x-a_{i+1})\cdots(x-a_n)}{(a_i-a_0)(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)}$$

for $i = 0, 1, 2, \ldots, n$.

(a) Verify that

$$f_i(a_j) = \begin{cases} 0, & \text{if } i \neq j; \\ 1, & \text{if } i = j. \end{cases}$$

(b) Show that $f_0, f_1, f_2, \ldots, f_n$ are linearly independent over $\mathbb{R}$.

(c) Let $b_0, b_1, b_2, \ldots, b_n$ be $n+1$ real numbers (not necessarily distinct). Let f be a polynomial of degree at most n such that $f(a_i) = b_i$ for $i = 0, 1, \ldots, n$. Express f as a linear combination of the f_i's over $\mathbb{R}$.