(1) (20 points) Construct an $\mathbb{R}$-linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ such that the range of $T =$ the null space of T.

(2) (20 points) In the real vector space $\mathcal{P}_2$ (the vector space of polynomials with real coefficients and of degree ≤ 2), define

$$\langle f,\ g\rangle = \int_0^1 f(x)g(x)\,dx.$$

(a) Show that $\langle -,\ -\rangle$ is an inner product.
(b) Use Gram-Schmidt process to transform $\{1, x,\ x^2\}$ into an orthonormal basis for $\mathcal{P}_2$.

(3) (20 points) Find a unitary matrix P such that

$$P^* \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} P$$

is diagonal. (P^* stands for the conjugate transpose of P.)

(4) (20 points) Let W_1 and W_2 be subspaces of a finite dimensional real vector space V. Suppose $W_1 \cap W_2 = \{O\}$ and $\dim W_1 + \dim W_2 = \dim V$. Show that any vector $v \in V$ can be written as

$$v = w_1 + w_2, \qquad \text{for some } w_1 \in W_1 \text{ and } w_2 \in W_2$$

in a unique way.

(5) (20 points) Find all the $a \in \mathbb{R}$ such that the matrix

$$\begin{pmatrix} 0 & 1 & a \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is similar to the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$