國立中正大學
111 學年度碩士班招生考試
試題
[第2節]

科目名稱	線性代數
系所組別	數學系

—作答注意事項—

※作答前請先核對「試題」、「試卷」與「准考證」之系所組別、科目名稱是否相符。

1. 預備鈴響時即可入場，但至考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。
2. 考試開始鈴響時，即可開始作答；考試結束鈴響畢，應即停止作答。
3. 入場後於考試開始 40 分鐘內不得離場。
4. 全部答題均須在試卷（答案卷）作答區內完成。
5. 試卷作答限用藍色或黑色筆（含鉛筆）書寫。
6. 試題須隨試卷繳還。

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NOTATION: In this test, all vector spaces are over $\mathbb{R}$. For a matrix $A \in M_{m\times n}$, let $L_A : \mathbb{R}^n \to \mathbb{R}^m$ denote left-multiplication transformation. Let $\mathbf{R}(L_A)$ denotes the range of L_A and $\mathbf{N}(A)$ denote the null space of L_A.

1. Let the matrix A be

$$A = \begin{pmatrix} 4 & 4 & 8 & 4 & 0 \\ 1 & 3 & 4 & 5 & 2 \\ 8 & 2 & 10 & 0 & 2 \\ 6 & 2 & 8 & 0 & 0 \end{pmatrix}.$$

(a) Compute the reduced row echelon form of matrix A. (10pts)

(b) From the answer of part (a), find a basis of $\mathbf{N}(L_A)$ and a basis of $\mathbf{R}(L_A)$. (10pts)

2. Let the matrix A be

$$B = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$

Determine whether B is diagonalizable. If B is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $B = PDP^{-1}$. (20pts)

3. Let $A \in M_{m\times n}(\mathbb{R})$.

(a) Show that $\mathbf{rank}(A^T A) = \mathbf{rank}(A)$. (10pts)

(b) Let the column vector $v \in \mathbb{R}^m$. If $\mathbf{rank}(A) = n$, find a vector w in $\mathbf{R}(L_A)$ which is the projection of v onto $\mathbf{R}(L_A)$. Note that $w \in \mathbf{R}(L_A)$ is the projection of v if $v \perp (v - w)$. (10pts)

4. Let V_1 and V_2 be subspaces of a vector space V having dimensions n_1 and n_2 respectively, where $n_2 \geq n_1$. Let n be the dimension of V.

(a) Prove that $\mathbf{dim}(V_1 \cap V_2) \leq n_1$. (6pts)

(b) Prove that $\mathbf{dim}(V_1 + V_2) \leq n_1 + n_2$. (6pts)

(c) Prove that $\mathbf{dim}(V_1 \cap V_2) \geq n_1 + n_2 - n$. (8pts)

5.

(a) Determine all possible value for $\mathbf{det}(A)$ when A is an orthogonal matrix. (10pts)

(b) Prove that if $A \in M_{n\times n}$ is skew-symmetric and n is odd, then A is not invertible. Give a counterexample when n is even. (10pts)